A Preference-based Semantics for CTD Reasoning

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Abstract. In [1] the authors developed a logical system based on the definition of a new non-classical connective $\otimes$ capturing the notion of reparative obligation. The system proved to be appropriate for handling well-known contrary-to-duty paradoxes but no model-theoretic semantics was presented. In this paper we fill the gap and define a suitable possible-world semantics for the system for which we can prove soundness and completeness. The semantics is a preference-based non-normal one extending and generalizing semantics for classical modal logics.

1 Introduction

One of the main research themes in deontic logic is about reasoning with contrary-to-duty (CTD) obligations [2]. In this perspective, it is widely acknowledged that the crisis of Standard Deontic Logic is historically and technically related to the formulation of some notorious paradoxes centering around the regulation of the violation of obligations.

The deontic logic literature on CTD reasoning is immense. However, two fundamental mainstreams have emerged as particularly interesting.

A first line of inquiry is mainly semantic-based. Moving from well-known studies on dyadic obligations, CTD reasoning is interpreted in settings with ideality or preference orderings on possible worlds or states [3]. The value of this approach is that the semantic structures involved are quite flexible: depending on the properties of the preference or ideality relation, different deontic logics can be obtained. This semantic approach has been fruitfully renewed in the '90 for example by [4,5], and most recently by works such as [6,7], which have confirmed the vitality of this line of inquiry.

The second mainstream is mostly proof-theoretic. Examples, among others, are various systems springing from Input/Output Logic [8,9] and the Gentzen system proposed in [1]. Both perspectives refer to the slogan "no logic of norms without attention to the normative systems in which they occur" [10], which draws inspiration from the pioneering works by [11] and [12]. This line of investigation is based on the intuition that any obligation can be explained in terms of a consequence relation of what is explicitly stated as obligatory in a normative system. While Input/Output approach mainly works by imposing some constraints

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on the manipulation of conditional norms, [1] is based on the introduction of the new non-classical binary operator $\otimes$: the reading of an expression like $a \otimes b$ is that a is primarily obligatory, but is this obligation is violated, the secondary obligation is b. The intuition behind this construction is that CTD obligations are a special kind of exception. For instance, the expression

$$\mathit{Invoice} \rightarrow \mathit{PayBy7days} \otimes \mathit{Pay5\%Interest} \otimes \mathit{Pay10\%Interest}$$

can be intuitively viewed as a compact representation of the following (where $\Rightarrow$ stands for any defeasible conditional)

$$\begin{aligned}&\mathit{Invoice} \Rightarrow \mathrm{OBL}\mathit{PayBy7days}\\ &\quad\mathrm{OBL}\mathit{PayBy7days}, \neg \mathit{PayBy7days} \Rightarrow \mathrm{OBL}\mathit{Pay5\%Interest}\\ &\qquad\mathrm{OBL}\mathit{Pay5\%Interest}, \neg \mathit{Pay5\%Interest} \Rightarrow \mathrm{OBL}\mathit{Pay10\%Interest}\end{aligned}$$

The logic for $\otimes$ proved to be flexible for several applied domains, such as in business process modeling [13], normative multi-agent systems [14], temporal deontic reasoning [15], and reasoning about different types of defeasible permission [16].

Nevertheless, no semantic model-theoretic analysis of the operator $\otimes$ has been so far provided. In this paper we fill the gap and define a suitable possible-world semantics for this operator. Such semantics is a preference-based non-normal one extending and generalizing neighbourhood frames for classical modal logics. In this perspective, our contribution may also offer useful insights for establishing connections between the two mentioned mainstreams on CTD reasoning.

The layout the paper is as follows. Section 2 presents the basic logical system for $\otimes$ by recalling some intuitions from [1] as well as by integrating the original logic with some new schemata. Section 3 defines a multi-preference neighbourhood semantics suitable for the system. Section 4 illustrates logic and semantics with a real-life scenario. Sections 5 and 6 provide, respectively, some characterization and completeness results. Further developments for future work are outlined in Section 7. Some conclusions end the paper.

2 The Logic of $\otimes$

Let us briefly summarize, adjust, and extend in this section the logic for the CTD operator $\otimes$ presented in [1].

The language consists of a countable set of atomic formulas. Well-formed-formulas are then defined using the usual Boolean connectives and the binary connective $\otimes$, which is intended to formalize CTD statements. The language of [1] is integrated here by adding the deontic operator O and P denoting, respectively, standard unary obligation and permission.[4]

[4] The original Gentzen system presented in [1] was based on a binary consequence relation $\vdash_{\mathsf{O}}$: an expression $\Gamma \vdash_{\mathsf{O}} a$ meant that, whenever the set of well-formed formulas Γ occurs, a is obligatory. We will not assign in the remainder a direct deontic meaning to $\vdash$, thus explicitly introducing the operator O. The permission operator P was not considered in the logic of [1].

The intended interpretation of an expression like $a \otimes b$ is that b is a deontic reparation of a or, more explicitly, that a is obligatory, but if this obligation is violated, then b becomes obligatory. Hence the operator $\otimes$ captures the combination of primary and CTD obligations into unique provisions.

The language is formally defined as follows:

Definition 1 (Language). *Let Prop* $= \{a, b, \dots\}$ *be a countable set of atomic propositions. Let* $X \in \{\mathsf{O}, \mathsf{P}\}$ *be the set of unary deontic operators, such that* $\mathsf{P} =_{def} \neg\mathsf{O}\neg$*, and* $\otimes$ *be the CTD binary operator.*

- *All atomic propositions are well formed formulas (wffs);*
- *If* a *and* b *are wffs, all Boolean expressions made using* a *and* b *are wffs;*
- *If* a *is a wff and* $\otimes$ *does not occur in* a*, then* Xa *is a wff;*
- *If* $a_1, \dots a_n$ *are Boolean expressions and* $\otimes$ *does not occur in* $a_1, \dots, a_n$*, then* $a_1 \otimes \cdots \otimes a_n$ *is a wff;*
- *Nothing else is a wff.*

The basic logical system for $\otimes$ consists of the following axiom schemata and inference rules.

$$\bigotimes_{i=1}^{n} a_i \equiv \left(\bigotimes_{i=1}^{k-1} a_i\right) \otimes \left(\bigotimes_{i=k+1}^{n} a_i\right) \quad \text{(where } a_j \equiv a_k,\ j < k\text{)} \qquad (\otimes\text{-contraction})$$

$$a \otimes (b \otimes c) \equiv (a \otimes b) \otimes c \qquad (\otimes\text{-associativity})$$

$$(a \otimes \neg a) \equiv \top \qquad (\otimes\text{-}\top 1)$$

$$(a \otimes \top) \equiv \top \qquad (\otimes\text{-}\top 2)$$

$$(\top \otimes a) \equiv \top \qquad (\otimes\text{-}\top 3)$$

$$(a \otimes \bot) \equiv \mathsf{O}a \qquad (\otimes\text{-}\bot 1)$$

$$(\bot \otimes a) \equiv \mathsf{O}a \qquad (\otimes\text{-}\bot 2)$$

$$\bigotimes_{i=0}^{n} a_i \otimes \bigotimes_{j=0}^{m} b_j \equiv \left(\bigotimes_{i=0}^{n} a_i\right) \otimes \bot \otimes \left(\bigotimes_{j=0}^{m} b_j\right) \qquad (\otimes\text{-}\bot 3)$$

A few comments are in order.

The first equivalence ($\otimes$-contraction) corresponds to duplication and contraction: for example, $a \otimes b \otimes a$ is equivalent to $a \otimes b$. Intuitively, if I'm obliged not to cause any damage, but if I cause any, then I have the obligation to compensate, and, if don't compensate, then I'm obliged not to cause any damage; this just means that my primary obligation is not to cause any damage and my secondary obligation is to compensate.

The meaning of ($\otimes$-associativity) is self-evident. Let us only remark that this implies that nested $\otimes$-formulas are meaningless in our language. This is in fact reflected in our previous Definition 1 when we have formally excluded from the set of wffs ⊗-expressions such as $\otimes a \otimes \neg(b \otimes c)$ and only accepted expressions like $\neg(a \otimes b \otimes c)$ or $(a \otimes b) \wedge (c \otimes d)$.

Schema (⊗-⊤1) says that, if my primary obligation is a and my secondary one is $\neg a$, although there is an order of preference, whatever I'm doing will be deontically acceptable (either ideal or sub-ideal). Again, if I'm obliged not to cause any damage, but, if I don't do that I'm obliged to cause damages, then I have a trivial normative provision. Hence $a \otimes \neg a$ is equivalent to $\top$. Analogously, (⊗-⊤2) and (⊗-⊤3) hold, as one of the two obligation is always satisfied.

Schemata (⊗-⊥1), (⊗-⊥2), and (⊗-⊥3) can be justified as follows. First of all, bear in mind that an expression like $\mathsf{O}a$ can be intuitively viewed as an ⊗-formula of length 1.[5] Indeed, if my primary obligation is a and my secondary one is $\bot$, since the latter cannot be satisfied in any possible world, then the expression $a \otimes \bot$ is equivalent to having a simple obligation $\mathsf{O}a$. Similar considerations apply to (⊗-⊥2) and (⊗-⊥3).

Introduction and elimination rules for ⊗ are as follows:

$$\frac{a \otimes (\bigotimes_{i=1}^{n} b) \otimes c \qquad \neg b_1 \wedge \cdots \wedge \neg b_n \rightarrow \bigotimes_{i=1}^{m} d_i}{a \otimes (\bigotimes_{i=1}^{n} b) \otimes (\bigotimes_{i=1}^{m} d_i)} \quad (\otimes\text{-I})$$

$$\frac{\bigotimes_{i=1}^{n} a_i \otimes b \otimes \bigotimes_{i=1}^{m} c_i \qquad \bigotimes_{i=1}^{n} a_i \otimes \neg b}{\bigotimes_{i=1}^{n} a_i \otimes \bigotimes_{i=1}^{m} c_i} \quad (\otimes\text{-E})$$

where $\bigotimes_{i=1}^{n} a_i \otimes b \otimes \bigotimes_{i=1}^{m} c_i \not\equiv \top$ and $\bigotimes_{i=1}^{n} a_i \otimes \neg b \not\equiv \top$.

Let us illustrate the introduction rule (⊗-I) by considering the well known 'dog, sign and fence' scenario [2]. The scenario contains the following four statements

1. There ought to be no dog;
2. If there is no dog, there ought to be no warning sign;
3. If there is a dog, there ought to be a warning sign;
4. If there is a dog and no warning sign, there ought to be a high fence.

The scenario is formalised as follows:

1. $\mathsf{O}\neg dog$
2. $\neg dog \rightarrow \mathsf{O}\neg sign$
3. $dog \rightarrow \mathsf{O}sign$
4. $dog \wedge \neg sign \rightarrow \mathsf{O}fence$

Clearly ⊗-I is applicable for 1. and 3. from which we derive

$$\neg dog \otimes sign.$$

At this point we can use the newly derived formula and 4. as the premise of ⊗-I to conclude

$$\neg dog \otimes sign.$$

As we have just seen the inference rule (⊗-I) generates chains of CTDs in order to deal iteratively with violations of compensatory obligations.

[5] We will see that it is not technically obvious how to capture this intuition.

The rule (⊗-E) operates in the opposite direction by removing in ⊗-formulas those propositions that are negated in other true ⊗-formulas. Here is a concrete example:

$$\frac{Pay\_Taxes \otimes Pay\_Interest \otimes Foreclosure \qquad Pay\_Taxes \otimes \neg Pay\_Interest}{Pay\_Taxes \otimes Foreclosure} \tag{1}$$

As extensively explained in [1], this shows that we have to distinguish between genuine normative conflicts from apparent ones. By normative conflict we mean any situation ruled by opposite norms and which results in an impossible state of affairs; or, in other words, a situation in which the normative content of all relevant norms cannot be fulfilled, ending inevitably in a violation that cannot be compensated by any other CTD.

3 Multi-preference Semantics

Let us introduce the semantic structures that we use to interpret ⊗-formulas. In fact, they are just an extension of neighbourhood frames for classical modal logics.

Definition 2. *A* multi-preference frame *is a tuple* $\mathcal{F} = \langle W, \mathcal{C} \rangle$ *where:*

- W *is a countable non empty set of worlds;*
- $\mathcal{C}$ *is a neighbourhood function with the following signature*

$$\mathcal{C} \colon W \mapsto 2^{((2^W)^n)}$$

such that for each $w \in W$*, for any ordered* n*-tuple* $\langle X_1, \dots, X_n \rangle$ *of subsets of* W *in* $\mathcal{C}_w$ *the following holds:*
  - *if* $i \neq j$*, then* $X_i \neq X_j$
  - *Or* $\bigcup_{1 \leq i \leq n} X_i = W$*.*

In general, a multi-preference frame is nothing but a structure where the standard neighbourhood function is replaced by a function that establishes an order between elements (i.e., sets of worlds) of each neighbourhood associated to every world. Figure 1 offers a pictorial representation of the intuition. The two conditions on sequences of sets of worlds are that there are no repetitions or that, if this is not the case, the union set of all sets in the sequence is W.

Given a formula $\bigotimes_{i=1}^{n} a_i$ we stipulate

$$\bigotimes_{i=1}^{n} a_i = \begin{cases} \mathsf{O}a_1 & n = 1 \\ a_1 \otimes \cdots \otimes a_n & n > 1 \end{cases}$$

The following definitions introduce the notion of redundancy and the operations of *zipping* and *s-zipping*, i.e., operations that, respectively, remove repetitions or redundancies occurring in ⊗-chains and in sequences of sets of worlds. Intuitively, these operations are necessary because, despite the fact the our language allows for building expressions like $a \otimes b \otimes a$ or $a \otimes \bot$, these last must be semantically evaluated using the sequences of sets of worlds $\langle \|a\|_V, \|b\|_V \rangle$ and $\langle \|a\|_V \rangle$ (see axiom schemata (⊗-contraction) and (⊗-⊥1)-(⊗-⊥3)).

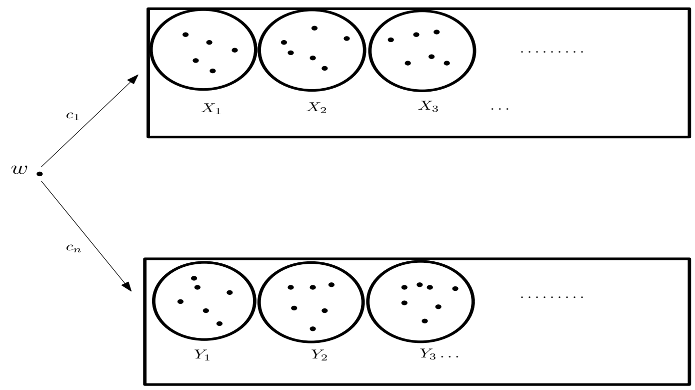

Fig. 1: Multi-preference basic structure: $X_1, X_2, X_3, \dots \subseteq W$ and $Y_1, Y_2, Y_3, \dots \subseteq W$

Definition 3. *A formula A is* redundant *iff $A = \bigotimes_{i=1}^{n} a_i$, $n > 1$ and*

- $\exists a_j, a_k$, $1 \leq j, k \leq n$ *such that* $a_j \equiv a_k$;
- $\exists a_j$, $1 \leq j \leq n$ *such that* $a_j \equiv \bot$.

Definition 4. *Let $A = \bigotimes_{i=1}^{n} a_i$ be any redundant formula. We say that the non-redundant B is* zipped from *A iff B is obtained from A by applying recursively the operations below:*

1. *If $n = 2$, i.e., $A = a_1 \otimes a_2$, and $a_1 \equiv a_2$, then B, the* zipped from, *is $\mathsf{O}a_1$;*
2. *Otherwise, if $n > 2$, then for $1 \leq k \leq n$, if (i) $a_j \equiv a_k$, for $j < k$, or (ii) $a_k \equiv \bot$, delete $\otimes a_k$ from the sequence.*

Let $X = \langle X_1, \dots, X_n \rangle$ such that $X_i \in 2^W$ $(1 \leq i \leq n)$. We analogously say that Y is s-zipped from *X iff Y is obtained from X by applying the operations below:*

1. *If $n = 2$ and $X_1 = X_2$, then its* s-zipped from *Y is $\langle X_1 \rangle$;*
2. *Otherwise, if $n > 2$, then for $1 \leq k \leq n$, if $X_j = X_k$, for $j < k$, or $X_k = \emptyset$, delete X_k from the sequence.*

Definition 5 (Models with sequences). *A model $\mathcal{M}$ is a couple $\langle \mathcal{F}, V \rangle$ where $\mathcal{F}$ is a frame and V is a valuation such that:*

- *for any non-redundant $\bigotimes_{i=1}^{n} a_i$, $\models_w^V \bigotimes_{i=1}^{n} a_i$ iff there is a $c_j \in \mathcal{C}_w$ such that $c_j = \langle \|a_1\|_V, \dots, \|a_n\|_V \rangle$;*
- *for any redundant $\bigotimes_{i=1}^{n} a_i$, $\models_w^V \bigotimes_{i=1}^{n} a_i$ iff*
  - *$\bigotimes_{f=1}^{k} a_f$ is zipped from $\bigotimes_{i=1}^{n} a_i$, and*
  - *$\models_w^V \bigotimes_{f=1}^{k} a_f$.*
- *$\models_w^V \mathsf{O}a$ iff there $c_l \in \mathcal{C}_w$ such that:*
  - *$c_l = \langle \|a_1\|_V, \dots, \|a_n\|_V \rangle$;*

- *for some* $k \leq n$, $X_k = \|a\|_V$*;*
- *for* $1 \leq j < k$, $w \notin X_j$*.*

Figure 2 pictorially illustrates the types of models used for evaluating $\otimes$-formulas. In fact, we use only finite sequences of worlds closed under s-zipping. A formula $\bigotimes_{i=1}^{n} a_i$ is true iff the corresponding appropriate finite sequence of sets of worlds (without redundancies) is in $\mathcal{C}_w$. Notice that the evaluation clause for $\mathsf{O}a$ works using sequences of length 1 or with longer sequences whenever a is the k's element of the $\otimes$-chain and the previous a_j are such that $w \notin \|a_j\|_V$, i.e., the previous obligations have been violated in w.

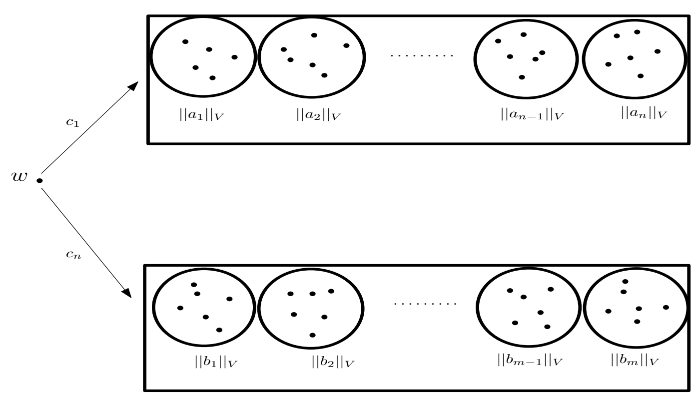

Fig. 2: Multi-preference models where finite sequences are used to evaluate the formulas $\bigotimes_{i=1}^{n} a_i$, ..., $\bigotimes_{i=1}^{m} b_i$

4 An Example

In [1] we showed that the $\otimes$-formalism is able to avoid most well-known CTD puzzles, such as Chisholm's and Forrester's paradoxes, Belzer's Reykjavik scenario and Makinson's Möbius strip example. These results can be trivially extended to the new variant of the logic presented here. Let us thus illustrate the the logic its semantics by considering a fresh deontic scenario, which seems to be problematic in most, if not all established formalisms dealing with CTD reasoning [17].

Suppose that a Privacy Act contains the following norms:

1. The collection of personal information is forbidden, unless acting on a court order authorising it.
2. The destruction of illegally collected personal information before accessing it is a defence against the illegal collection of the personal information.
3. The collection of medical information is forbidden, unless the entity collecting the medical information is permitted to collect personal information.

In addition, the Act specifies what personal information and medical information are, and they turn out to be disjoint.

Suppose an entity, subject to the Act, collects some personal information without being permitted to do so; at the same time they collect medical information. The entity recognises that they illegally collected personal information (i.e., they collected the information without being authorised to do so by a Court Order) and decides to remediate the illegal collection by destroying the information before being accessing it. Are they compliant with the above privacy act? Given that the personal information was destroyed the entity was excused from the violation of the first section (illegal collection of personal information). However, even if the entity was excused from the illegal collection, they were never entitled (i.e., permitted) to collect personal information[6], consequently they were not permitted to collect medical information; thus the prohibition of collecting medical information was in force. Accordingly, the collection of medical information violates the norm forbidding such an activity.

The logical structure of the act can be represented by the following norms:

1. a is forbidden, its violation is compensated by b.
2. a is permitted given c.
3. d is forbidden.
4. If a is permitted, so is d.

Let us consider the situations compliant with the above set of norms. Clearly, if c does not hold, then we have that the prohibitions of a and d are in force. Therefore, a situation where $\neg a$, $\neg c$, and d hold is fully compliant (irrespective whether b holds or not). If c holds, then the permission of a derogates the prohibition of a (situations with either a or $\neg a$ are compliant with he first two norms); in addition, the permission of a allows us to derogate the prohibition of d. Accordingly, situations with either d or $\neg d$ comply with the third norm. Let us go back to scenarios where c does not hold, and let us suppose that we have a. This means that the prohibition of a has been violated; nevertheless the set of norms allows us to recover from such violation by b. However, to have a violation we have to have either an obligation or a prohibition that has been violated: in this case the prohibition of a. Given that prohibition of a and permission of a are mutually incompatible, we must have, to maintain a consistent situation, that a is not permitted. But if a was not permitted, d is not permitted either; actually, according to the third norm, d is forbidden.

To sum up, a scenario where $\neg c$, a, b and $\neg d$ hold is still compliant (even if to a lesser degree given the compensated violation of the prohibition of a). In any case, no situation where both $\neg c$ and d hold is compliant.

The scenario can be reconstructed in our logic by meeting the desiderata[7]:

[6]If they were permitted to collect personal information, then the collection would have not been illegal, and they did not have to destroy it.

[7]The scenario uses strong permissions, i.e., the permissions derogating the obligations to the contrary. To accomplish this we have to specify that 2 overrides 1, and 4 overrides 3. This explains the Boolean structure of 1 and 3.The focus of this paper is not how

1. $\neg c \rightarrow (\neg a \otimes b)$;
2. $c \rightarrow \mathsf{P}a$;
3. $\mathsf{O}\neg a \rightarrow \mathsf{O}\neg d$;
4. $\mathsf{P}a \rightarrow \mathsf{P}d$.

If $\neg c$, a, b and $\neg d$, then the deontic provision from 1 is sub-ideally satisfied by b; $\mathsf{P}a$ and $\mathsf{P}d$ cannot be obtained; $\mathsf{O}\neg d$ is obtained in 3 and satisfied. If $\neg c$ and d, then $\mathsf{O}\neg d$ is neither satisfied nor compensated. Let us also analyze the scenario and these cases semantically:

Let us consider a model $\mathcal{M}$ with a world w such that

- $w \notin \|c\|_V$, $w \notin \|d\|_V$, $w \in \|a\|_V$ and $w \in \|b\|_V$; and
- $\mathcal{C}_w = \{\langle \|\neg a\|_V, \|b\|_V \rangle, \langle \|\neg d\|_V \rangle\}$.

It is easy to verify that 1.–4. above are all true at w. Let us see the reasons why they are true. 1.) Since the sequence $\langle \|\neg a\|_V, \|b\|_V \rangle \in \mathcal{C}_w$, from which $\models^V_w \neg a \otimes b$ follows. 2.) trivially, since $w \notin \|c\|_V$. 3.) Similarly to 1. the sequence $\langle \|\neg d\|_V \rangle \in \mathcal{C}_w$; notice that we have $\models^V_w \mathsf{O}\neg s$ given that $\langle \|\neg a\|_V, \|b\|_V \rangle \in \mathcal{C}_w$, in addition $w \in \|a\|_V$, from which it follows $\models^V_w \mathsf{O}\neg d$. 4.) As we have already seen, $\models^V_w \mathsf{O}\neg a$, thus from the definition of P we get $\not\models^V_w \mathsf{P}a$; similarly for $\not\models^V_w \mathsf{P}d$.

In the scenario corresponding to w we are (weakly) compliant because $\mathsf{O}\neg a$ is violated (i.e., $w \in \|a\|_V$), but $\mathsf{O}b$ (which compensates $\mathsf{O}\neg a$) is complied with. Also $\mathsf{O}\neg d$ is complied with. Consider now a possible world y which is like w but $y \in \|d\|_V$. In y we are not compliant: we have a violation of $\mathsf{O}\neg d$ (which is not compensable).

5 Characterization Results

Let us consider the following inference rule:

$$\frac{\vdash \bigwedge_{i=1}^{n}(a_i \equiv b_i)}{\vdash (\bigotimes_{i=1}^{n} a_i) \equiv (\bigotimes_{i=1}^{n} b_i)} \qquad (\otimes\text{-RE})$$

It should be intuitively clear that ($\otimes$-RE) generalizes for $\otimes$-formulas the weakest inference rule for modal logics, i.e., the closure of $\Box$ (here O) under logical equivalence [18]:

$$\frac{\vdash a \equiv b}{\vdash \mathsf{O}a \equiv \mathsf{O}b} \qquad \text{(RE)}$$

Lemma 1. *($\otimes$-RE) and (RE) hold in the class of all multi-preference frames, i.e., on the class of all multi-preference frames,*

- *if* $\bigwedge_{i=1}^{n}(a_i \equiv b_i)$ *is valid, then* $(\bigotimes_{i=1}^{n} a_i) \equiv (\bigotimes_{i=1}^{n} b_i)$ *is valid ($\otimes$-RE);*
- *if* $a \equiv b$ *is valid, then* $\mathsf{O}a \equiv \mathsf{O}b$ *is valid (RE).*

to implement defeasibility, thus we just adopt the simplest procedure to handle this aspect.

Proof (Sketch). The result for ($\otimes$-RE) trivially follows from the fact the valuation clause for any $\otimes$-formula $\bigotimes_{i=1}^{n} a_i$, at any world w and with any valuation V, requires the existence of a sequence $c \in \mathcal{C}_w$ of truth sets $\langle \|a_1\|_V, \dots, \|a_n\|_V \rangle$. (RE), too, trivially follows as a special case of ($\otimes$-RE): indeed, by stipulation $\bigotimes_{i=1}^{n} a_i = \mathsf{O}a_i$ when $n = 1$ and, semantically, $\mathsf{O}a$ is true in w iff there is a finite sequence $\langle X_1, \dots, X_n \rangle$, where $n \geq 1$, such that $\|a\|_V = X_k$ and $w \notin X_1, \dots, X_{k-1}$.

Also ($\otimes$-contraction), ($\otimes$-associativity), and ($\otimes$-$\bot$1)-($\otimes$-$\bot$3) hold in general:

Lemma 2. *($\otimes$-contraction), ($\otimes$-associativity), and ($\otimes$-$\bot$1)-($\otimes$-$\bot$3) are valid in the class of all multi-preference frames.*

Proof (Sketch). Consider ($\otimes$-contraction). First of all, remember that, by construction, all sequences $\langle X_1, \dots, X_n \rangle$ for which we do not have that $\bigcup_{i=1}^{n} X_i \neq W$ are such that $X_k \neq X_j$, $\forall k, j \in \{1, \dots, n\}$. Since sequences in frames are closed under the operation of s-zipping (see Definitions 4 and 5, i.e., the sequences used to evaluate $\otimes$-formulas do not contain repetitions), thus every redundant $\otimes$-formula is uniquely evaluated by one sequence without repetitions.

($\otimes$-associativity) holds for similar reasons: just consider how $\otimes$-formulas are recursively evaluated.

Finally, consider ($\otimes$-$\bot$1)-($\otimes$-$\bot$3). Indeed, all $\otimes$-formulas $\bigotimes_{i=1}^{n} a_i$ where at least one $a_k \equiv \bot$, $1 \leq k \leq n$, is redundant and so is evaluated by considering a sequence s-zipped from a sequence $\langle \|a_1\|_V, \dots, \|a_k\|_V = \emptyset, \dots, \|a_n\|_V \rangle$.

Finally, schemata ($\otimes$-$\top$1)-($\otimes$-$\top$3) hold, too:

Lemma 3. *($\otimes$-$\top$1)-($\otimes$-$\top$3) are valid in the class of all multi-preference frames.*

Proof (Sketch). The proof follows from considering sequences $\langle X_1, \dots, X_n \rangle$ where $\bigcup_{i=1}^{n} X_i = W$. This makes trivially valid ($\otimes$-$\top$1); ($\otimes$-$\top$2) and ($\otimes$-$\top$3) are also valid because $\|a\|_V \cup W = W$ for any formula a.

Notice that the following axiom schema holds, too:

$$((\bigotimes_{i=1}^{n} b_i) \otimes c \otimes \bigotimes_{j=1}^{n} d_j)) \wedge (\bigwedge_{i=1}^{n} \neg b_i)) \rightarrow \mathsf{O}c \qquad (\mathsf{O}\text{-Detachment})$$

Lemma 4. *(O-Detachment) is valid in the class of all multi-preference frames.*

Proof (Sketch). The proof trivially follows from the definition of the operator O and the valuation clause for it.

The class of all multi-preference frames cannot validate introduction and elimination rules for $\otimes$, which require extra semantic conditions.

Conditions for characterizing ($\otimes$-I) are as follows:

Definition 6. *Let $\mathcal{F} = \langle W, \mathcal{C} \rangle$ be a frame. We say that $\mathcal{F}$ is* $\otimes$-expanded *iff for any $w \in W$ and $c_i, c_j \in \mathcal{C}_w$ such that $c_i = \langle X_1, \dots, X_n \rangle$, $c_j = \langle Y_1, \dots Y_m \rangle$, $\forall l \colon 1 \leq k < l < f \leq n$, $w \in W - X_l$, then there exists $c' \in \mathcal{C}_w$ such that c' is s-zipped from $\langle X_1, \dots, X_k, Y_1, \dots, Y_m \rangle$.*

Lemma 5. *($\otimes$-I) holds in the class of expanded multi-preference frames, i.e., on the class of expanded multi-preference frames, if*

$$a \otimes (\bigotimes_{i=1}^{n} b) \otimes c \tag{2}$$

$$\neg b_1, \ldots, \neg b_n \to \bigotimes_{i=1}^{m} d_i \tag{3}$$

are valid, then

$$a \otimes (\bigotimes_{i=1}^{n} b) \otimes (\bigotimes_{i=1}^{m} d_i) \tag{4}$$

is valid.

Proof (Sketch). By reductio, suppose that (2) and (3) are valid and that (4) is not. This means that there is a world w such that

1. $w \in \|\neg b_1\|_V \cap \ldots \cap \|\neg b_n\|_V$,
2. $\exists c_i \in \mathcal{C}_w$ such that c_i is s-zipped from $\langle \|a\|_V, \|b_1\|_V, \ldots, \|b_n\|_V, \|c\|_V \rangle$, and
3. $\exists c_j \in \mathcal{C}_w$ such that c_j is s-zipped from $\langle \|d_1\|_V, \ldots, \|d_m\|_V \rangle$, but
4. there is no $c_k \in \mathcal{C}_w$ such that c_k is s-zipped from $\langle \|a\|_V, \|b_1\|_V, \ldots, \|b_n\|_V, \|d_1\|_V, \ldots, \|d_m\|_V \rangle$.

From 2, it is clear that there is a subsequence of c_i' which is s-zipped from $\langle \|a\|_V, \|b_1\|_V, \ldots, \|b_n\|_V \rangle$. We concatenate c_i' and c_j creating the sequence $c_i'c_j$. If $c_i'c_j$ is s-zipped from itself we are done, since, in conjunction with 1, 2, 3 and the fact that the from is $\otimes$-expanded give us that $c_i'cj \in \mathcal{C}_w$ and $c_i'c_j$ is s-zipped from $\langle \|a\|_V, \|b_1\|_V, \ldots, \|b_n\|_V, \|d_1\|_V, \ldots, \|d_m\|_V \rangle$. Contradiction. If $c_i'c_j$ is not s-zipped form itself, this means that there some elements of c_j which appear in c_i'. We create c_j' by remove such elements from c_j and we concatenate c_i' and c_j', obtaining the sequence $c_i'c_j'$. Again, by 1, 2, 3 and the fact that the frame is $\otimes$-expanded we have that $c_i'c_j' \in \mathcal{C}_w$ and it is mundane to verify that $c_i'c_j'$ is s-zipped from $\langle \|a\|_V, \|b_1\|_V, \ldots, \|b_n\|_V, \|d_1\|_V, \ldots, \|d_m\|_V \rangle$. Contradiction again.

Before giving the semantic conditions for ($\otimes$-E) we introduce some auxiliary concepts. Trivially any sequence induces a relation. Accordingly, for every world $w \in W$ and for every $c_i \in \mathcal{C}_w$ where $c_i = \langle X_1, \ldots, X_n \rangle$ we have a relation $\#_i$ such that $(X_j, X_{j+1}) \in \#_i$ for $1 \leq j \leq n$.

Definition 7. *Let $\mathcal{F} = \langle W, \mathcal{C} \rangle$ be a frame. We say that $\mathcal{F}$ is* $\otimes$-contracted *iff for any $w \in W$ and $c_i, c_j \in \mathcal{C}_w$ such that $c_i = \langle X_1, \ldots, X_n \rangle$, $\bigcup_{1 \leq i \leq n} X_i \neq W$, $c_j = \langle X_1, \ldots, X_{k-1}, W - X_k \rangle$, $(W - X_k) \cup \bigcup_{1 \leq i \leq k-1} X_i \neq W$ then*

$$\{c' \colon \#' = (c_i - \{(X_{k-1}, X_k), (X_k, X_{k+1})\}) \cup \\ \cup \{(X_{k+s}, X_k), (X_k, X_{k+s+1})\}_{1 \leq s < n-k}\} \subset \mathcal{C}_w$$

Lemma 6. *($\otimes$-E) is characterized by the class of $\otimes$-contracted multi-preference frames.*

Proof (Sketch). Suppose by reduction that it does not hold. Thus there is a valuation V and a world w such that

(a) $\models^V_w \bigotimes_{i=1}^n a_i \otimes b \otimes \bigotimes_{i=1}^m c_i$;
(b) $\models^V_w \bigotimes_{i=1}^n a_i \otimes \neg b \otimes \bigotimes_{i=1}^f d_i$; but
(c) $\not\models^V_w \bigotimes_{i=1}^n a_i \otimes \bigotimes_{i=1}^m c_i$

(c) implies that there is no $c' \in \mathcal{C}_w$ which is s-zipped from $\langle \|a_1\|_V, \ldots, \|a_n\|_V, \|c_1\|_V, \ldots, \|c_m\|_V\rangle$. From (a) we know that there is a $c_i \in \mathcal{C}_w$, such that $c_i = \langle X_1, \ldots, X_k, X, Z_1, \ldots, Z_s\rangle$ which is s-zipped from $\langle \|a_1\|_V, \ldots, \|a_n\|_V, \|b\|_V, \|c_1\|_V, \ldots, \|c_m\|_V\rangle$; from (b) we know that there is a $c_j \in \mathcal{C}_w$ such that $c_j = \langle \|X_1\|_V, \ldots, \|X_k\|_V, W - \|X\|_V\rangle$ such that c_j is s-zipped from $\langle \|a_1\|_V, \ldots, \|a_n\|_V, \|\neg b\|\rangle$; since the $\bigotimes_{i=1}^n a_i \otimes \neg b \not\equiv \top$ we infer that b does not appear in $\bigotimes_{i=1}^n a_i$ and then $X = \|b\|_V$.

We consider two cases (i) b does not appear in $\bigotimes_{i=1}^m c_i$, and (ii) b appears in $\bigotimes_{i=1}^m c_i$. (i) is trivial since given that b does not appear, $X \neq Z_u$ $(1 \leq u \leq s)$, and thus $c'_i = \langle X_1, \ldots, X_k, Z_1, \ldots, Z_s\rangle$ is in $\mathcal{C}_w$ given that the frame is $\otimes$-contracted and c'_i is s-zipped from $\langle \|a_1\|_V, \ldots, \|a_n\|_V, \|c_1\|_V, \ldots, \|c_m\|_V\rangle$. Contradiction.

6 The Minimal System $\mathsf{E}^{\otimes}$

Let us consider the minimal system $\mathsf{E}^{\otimes}$, which is given by adding to the classical propositional calculus the modal schemata $\otimes$-contraction, $\otimes$-associativity, $\otimes$-$\top$1, $\otimes$-$\top$2, $\otimes$-$\top$3, $\otimes$-$\bot$1, $\otimes$-$\bot$2, $\otimes$-$\bot$3, O-Detachment, and it is closed under the rules $\otimes$-**RE**, **RE**, and **MP**.

Definition 8 ($\mathsf{E}^{\otimes}$-Canonical Models). *A multi-preference model with sequences $\mathcal{M} := \langle W, \mathcal{C}, V\rangle$ is a canonical model for $\mathsf{E}^{\otimes}$ if and only if:*

1. *$W := \{w \mid w$ is $\mathsf{E}^{\otimes}$-maximal$\}$*
2. *for any propositional letter $p \in Prop$, $\|p\|_V := |p|_{\mathsf{E}^{\otimes}}$, where $|p|_{\mathsf{E}^{\otimes}} := \{w \in W \mid p \in w\}$*
3. *Let $\mathcal{C} := \bigcup_{w \in W} \mathcal{C}_w$, where for each $w \in W$, $\mathcal{C}_w := \{\langle \|A_1\|_V, \ldots, \|A_n\|_V\rangle \mid A_1 \otimes \cdots \otimes A_n \in w\} \cup \{\langle A\rangle_V \mid \mathsf{O}A \in w\}$, where each A_i is a meta-variable for a Boolean formula and $A_1 \otimes \cdots \otimes A_n$ is* zipped.

Notice that since each $\mathcal{C}_w$ contains only ordered sequences of truth sets obtained by *zipped* formulas, the following condition holds true: for any ordered n-tuple $\langle X_1, \ldots, X_n\rangle$ of subsets of W in $\mathcal{C}_w$ m if $i \neq j$, then $X_i \neq X_j$. Moreover, $\mathcal{C}$ contains only s-zipped sequences.

Lemma 7 (Truth Lemma). *$A \in w$ if and only if $\models_w A$.*

Proof (Sketch). Given the construction of the canonical model, this proof is quite straightforward and it can be given by induction on the length of a formula A. We consider only the modal cases.

Assume A has the form $a_1 \otimes \cdots \otimes a_n$ and that is redundant (clearly the case for non redundant formulas is easier and does not need to be considered here). Suppose $a_i \otimes \cdots \otimes a_n \in w$. Then, by Axiom $\otimes$-contraction and $\otimes$-$\perp$3, we have that the formula $b_1 \otimes \cdots \otimes b_j$, the *zipped* form of A, is also in w. By definition of canonical model we have that there is a sequence $\langle \|b_1\|_V, \ldots, \|b_j\|_V \rangle \in \mathcal{C}_w$. Following from the semantic clauses given to evaluate $\otimes$-formulas, it holds that $\models^w_V a_1 \otimes \ldots \otimes a_n$.

Now suppose that $\models^w_V a_1 \otimes \cdots \otimes a_n$. By definition, there is a zipped formula $b_1 \otimes \cdots \otimes b_j$ such that $\models^w_V b_1 \otimes \cdots \otimes b_j$. Thus, C_w contains an ordered j-tuple $\langle \|b_1\|_V, \ldots, \|b_j\|_V \rangle$. By definition of $\mathcal{C}_w$ it follows that $b_1 \otimes \cdots \otimes b_j \in w$ and by the axioms $\otimes$-contraction and $\otimes$-$\perp$3, all the *unzipped* forms of $b_1 \otimes \cdots \otimes b_j$ are also in w, including $a_1 \otimes \cdots \otimes a_n$.

If, on the other hand, A has the form $\mathsf{O}b$ and $\mathsf{O}b \in w$, then $\|b\|_V \in \mathcal{C}_w$ and, by definition $\models^w_V \mathsf{O}b$. Conversely, if $\models^w_V \mathsf{O}b$, then there is an s-zipped sequence $\langle \|c_1\|_V, \ldots, \|c_n\|_V, \|b\|_V, \|d_1\|_V, \ldots, \|d_m\|_V \rangle \in \mathcal{C}_w$ and for $1 \leq i \leq n$, $w \notin \|c\|_i$. Thus, since any c_i is Boolean and w is maximal, $\neg c_1, \ldots, \neg c_n \in w$. Moreover $\bigotimes_{i=1}^n c_i \otimes b \otimes \bigotimes_{j=1}^m d_j \in w$. Hence by the O-Detachment axiom and **MP**, $\mathsf{O}b \in w$.

Corollary 1. *The logic $\mathsf{E}^{\otimes}$ is sound and complete with respect to the class of multi-preference frames with zipped sequences.*

7 Semantics with Multi-relational Frames

In this section we briefly outline a possible extension of the framework presented above. It consists in working with structures validating stronger modal inference rules and schemata, in particular

$$\mathsf{O}\top \qquad (\mathbf{N})$$

$$\frac{\vdash \bigwedge_{i=1}^n (a_i \to b_i)}{\vdash (\bigotimes_{i=1}^n a_i) \to (\bigotimes_{i=1}^n b_i)} \qquad (\otimes\text{-RM})$$

and

$$\frac{a \to b}{\mathsf{O}a \to \mathsf{O}b} \qquad (\mathsf{O}\text{-RM})$$

This means moving to a class of logic called N-monotonic [19]. Hence, we can employ a simple generalization of Kripke semantics with a countable set of accessibility relations [20]. This basic semantic setting is here enhanced by adding preferences over sets of worlds.[8]

[8] See [21] for a similar construction, which is however used for different purposes.

7.1 Multi Relational Frames with Sequences

Let $\mathsf{NM}^{\otimes}$ be the system obtained by adding $\mathbf{N}$, ($\otimes$-RM), and (O-RM) to $\mathsf{E}^{\otimes}$.

Definition 9. *A frame is a tuple $\mathcal{F} = \langle W, \mathcal{R}, f, \mathcal{C}\rangle$ where:*

- *W is a possibly infinite set of worlds;*
- *$\mathcal{R}$ is a countable set of binary relations over W;*
- *Let f be a function $f : W \longrightarrow \mathcal{P}(\mathcal{P}(W))$ be a function assigning to each world $w \in W$ a set of finite ordered n-tuples of subsets of W, i.e., for each $w \in W$, $\mathcal{C}_w := \{\langle X_1, \dots, X_i\rangle, \langle Y_1, \dots Y_j\rangle, \dots\}$.*
- *$\mathcal{C} := \bigcup_{w\in W} \mathcal{C}_w$.*

Notation and abbreviations. Let $R_i(w) := \{v \in W \mid wR_iv\}$, $\|A\|_V := \{w \in W \mid \ \models^V_w A\}$.

Definition 10 (Multi relational models with sequences).
A model $\mathcal{M}$ is a couple $\langle \mathcal{F}, V\rangle$ where V is a valuation such that:

- *$\models^V_w a_1 \otimes \cdots \otimes a_n$ where $a_1 \otimes \cdots \otimes a_n$ is zipped, iff there is a finite sequence $\langle X_1, \dots, X_n\rangle \in \mathcal{C}_w$ such that for $1 \le j \le n$:*
  - *$X_j = \|a_j\|_V$*
  - *For each j, $1 \le j \le n$, there is a relation R_i such that $R_i(w) \subseteq X_j$*
- *$\models^V_w a_1 \otimes \cdots \otimes a_n$ where $a_1 \otimes \cdots \otimes a_n$ is redundant, iff $\models^V_w b_1 \otimes \cdots \otimes b_j$, where $b_1 \otimes \cdots \otimes b_j$ is its zipped form.*
- *$\models^V_w \mathsf{O}A$ iff there is a finite sequence $\langle X_1 \dots X_n\rangle \in \mathcal{C}_w$ such that for $1 \le j \le n$:*
  - *$X_j = \|a_j\|_V$ for some proposition a_j;*
  - *$R_m(w) \subseteq X_j$ for some $R_m \in \mathcal{R}$*
  - *for some $k \le n$, $X_k = \|A\|_V$*
  - *for $1 \le j < k$, $w \notin X_j$.*

7.2 Completeness Sketch

Definition 11 ($\mathsf{NM}^{\otimes}$-Canonical Models). *Let $\mathcal{M} := \langle W, \mathcal{R}, \mathcal{C}, V\rangle$ be a multi-relational model with sequences. $\mathcal{M}$ is a canonical model for $\mathsf{NM}^{\otimes}$ if and only if:*

1. *$W := \{w \mid w$ is $\mathsf{NM}^{\otimes}$-maximal$\}$*
2. *for any propositional letter $p \in Prop$, $\|p\|_V := |p|_{\mathsf{NM}^{\otimes}}$, where $|p|_{\mathsf{NM}^{\otimes}} := \{w \in W \mid p \in w\}$*
3. *For each $w \in W$ and each natural number n, let $\mathcal{C}_w := \{\langle \|a_1\|_V, \dots, \|a_n\|_V\rangle \mid a_1 \otimes \cdots \otimes a_n \in w$ and it is zipped $\} \cup \{\langle A\rangle_V \mid \mathsf{O}A \in w\}$, where a_i is Boolean. Let $\mathcal{C} := \bigcup_{w\in W} \mathcal{C}_w$.*
4. *$\mathcal{R} := \{R^\star_{A_j} \mid A_j \in Fma(\mathcal{L})\} \cup \{R^i_{A_j} \mid A_j \in Fma(\mathcal{L})\}_{i\in\mathbb{N}}$*
   (a) For each $w \in W$, $R^\star_{A_j}(w) := \|A_j\|_V$ iff there is some $B_1 \otimes \cdots \otimes B_n \in w$ such that $B_i = A_j$;

(b) For each $w \in W$, consider any subsequence $\langle \|A_1\|_V, \ldots, \|A_k\|_V \rangle$ such that $\langle \|A_1\|_V, \ldots, \|A_n\|_V \rangle \in \mathcal{C}_w$ and $k \leq n$ and $\mathsf{O}A_k \in w$, and $\neg A_i \in w$ for $1 \leq i \leq k-1$. Let $\mathcal{C}_w^1$, $\mathcal{C}_w^2, \ldots$ be an enumeration of such subsequences. Then for each $\mathcal{C}_w^i$ set the following: $R^i_{A_k}(w) := \|A_k\|$.

Lemma 8 (Truth Lemma). *$A \in w$ if and only if $\models_w A$.*

Proof. Again, let us consider only the modal cases. Assume a redundant formula $a_1 \otimes \cdots \otimes a_n \in w$. Then its zipped form $b_1 \otimes \cdots \otimes b_j \in w$ by the axioms $\otimes$-contraction and $\otimes$-$\perp$3 and modus ponens; also, by construction, the n-tuple $\langle \|b_1\|, \ldots, \|b_j\| \rangle \in \mathcal{C}_w$ and there are relations $R^\star_{b_1}, \ldots, R^\star_{b_j}$ such that for each i, $R^\star_{b_i}(w) = \|b_i\|$. Thus, $\models^w_V a_1 \otimes \cdots \otimes a_n$, its expanded form.

Analogously, suppose $\models^w_V a_1 \otimes \cdots \otimes a_n$. Then $\models^w_V b_1 \otimes \cdots \otimes b_j$, i.e., its zipped form. This means that there are relations $R_i, \ldots, R_j$ such that $R_i(w) \subseteq \|b_i\|$ for each i and $\langle \|b_1\|, \ldots, \|b_j\| \rangle \in \mathcal{C}_w$. By definition of $\mathcal{C}_w$, it holds that $b_1 \otimes \cdots \otimes b_j \in w$ and by the axioms $\otimes$-contraction and $\otimes$-$\perp$3 and modus ponens there are also all its expanded forms, including $a_1 \otimes \cdots \otimes a_n$.

Suppose $\mathsf{O}A$ is in w. By definition, $\mathcal{C}_w$ contains the 1-tuple $\langle \|A\| \rangle$ and for some i, $R^i_A(w) = \|A\|$ and hence $\models^w_V \mathsf{O}A$.

Conversely, if $\models^w_V \mathsf{O}A$, then by definition there is some finite n-tuple $\langle \|b_1\| \ldots \|b_n\| \rangle \in \mathcal{C}_w$ such that for $1 \leq j \leq n$:

- $R_m(w) \subseteq \|b_j\|$ for some $R_m \in \mathcal{R}$
- for some $k \leq n$, $\|b_k\| = \|A\|_V$
- for $1 \leq j < k$, $w \notin \|b_j\|$.

Thus, the formula $b_1 \otimes \cdots \otimes b_k \otimes \cdots \otimes b_n \in w$ by definition of $\mathcal{C}_w$, $\neg b_1, \ldots, \neg b_i \in w$ and by the O-Detachment axiom $\mathsf{O}b_i \in w$. But $b_i \equiv A$, so by the **RM** rule and modus ponens, $\mathsf{O}A \in w$ too.

8 Conclusion

This paper offered a semantic study of the $\otimes$ operator originally introduced in [1]. We showed that a suitable logical system can be characterized in a class of structures extending neighbourhood frames with sequences of sets of worlds. In this perspective, our contribution may offer useful insights for establishing connections between the proof-theoretic and model theoretic approaches to CTD reasoning.

A number of open research issues are left for future work. Among others, we aim at going beyond basic completeness results with multi-preference structures for the classical case by considering introduction or elimination rules for $\otimes$, for which we only presented characterization results. Second, an extensive investigation should be done when we move to logics closed under logical implication (see Section 7). Finally, we expect to enrich the language and allow for nesting of $\otimes$-expressions, thus having formulas like $\neg(a \otimes b) \otimes c$; although we argued in [1] that the meaning of those formulas is not clear, they pose anyway interesting technical problems.

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